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Explicit solutions for the Harry Dym equation

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Abstract. The well known link between Korteweg–de Vries and HD (Harry Dym) equations is improved in this paper by the introduction of a suitable spectral parameter in the corresponding Bäcklund relationship. This additional spectral parameter allows the complete transfer of the group theoretical structure between the two equations. As an application we give new solutions of the HD on finite intervals. Several plots are included. The method of solution is also applied to the Kawamoto equation.

1. Introduction

The Harry Dym (HD) equation [15] for $\varrho = \varrho(t, \xi)$ is

$$\varrho_t = \varrho^3 \varrho_{\xi\xi\xi}. \tag{1.1}$$

Seemingly [12], this nonlinear PDE was found by Harry Dym while trying to transfer some results about isospectral flows to the string equation. The relationship between the HD and the classical string problem, with variable elastic parameter, was pointed out again in 1979 by Sabatier [24]. Since then, a wealth of information has been gathered on this equation: its bi-Hamiltonian formulation [19], its complete integrability [26] together with infinitely many conservation laws [27], and the applicability of the spectral gradient method [16, 17]. Direct links were found between either the KdV and HD [22, 11], or the mKdV and HD [14]. A relationship between the HD, with an additional potential term, and the KdV was given earlier [3].

There seems to be considerable interest in explicit solutions for the HD equation. However, such solutions are difficult to obtain because this equation is not a quasilinear equation (nor an equation with a constant *separant* in the terminology of [13]). As a consequence, many of the well known structural properties, which are important for obtaining explicit solutions, do fail; in particular the method for finding Bäcklund transformations via the prolongation scheme [17] (see also [18]). The structural reason for the exceptional position of the HD equation is that it has two scaling symmetries which makes it different from the folklore equations in 1 + 1 dimensions. This is also the reason why some other established methods fail, such as the validity of the Tu theorem [25] (see also [13, p 269] for a more careful version of this theorem), or have to be modified, such as the exact Painlevé test ([29] or [21]).

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Today, up to a certain order there are complete catalogues of equations with such exceptional behaviour [10, 28] and we are certain that the results exhibited in this paper can also be transferred, after slight modifications, to these equations. As one example we present the Kawamoto equation [14] at the end of this paper.

The essential tool for obtaining explicit solutions for these equations is the well known method of reciprocal transformation (see [11], [22] or [23]). By use of this method, and by the introduction of an additional spectral parameter, which is intrinsically related to the reciprocal transformation, we construct explicit solutions in such a way that they can be easily plotted. These plots then open new avenues for the study of qualitative behaviour of soliton interaction.

2. Principal result

First, a simple method will be described to obtain solutions of the HD equation from known solutions of the KdV equation

$$u_t = u_{xxx} + 6uu_x. \tag{2.1}$$

Let $u = u(x,t)$ be such a solution, then define φ to be the solution of the following Riccati equation

$$4u - 2\varphi_x + \varphi^2 = c^2 \tag{2.2}$$

where c is some spectral parameter. Then choose s to be a solution of

$$s_x = s\varphi \tag{2.3}$$

where, for later convenience, the constant of integration is fixed in such a way that

$$\int_{-\infty}^{+\infty} s(x,t)\,\mathrm{d}x = 2\sqrt{c}. \tag{2.4}$$

This choice of integration constant corresponds to a normalization of s such that it has the same mass (i.e. integral from $-\infty$ to $+\infty$) as a single soliton. Now, use the well known *reciprocal transformation*

$$\xi := \int_{-\infty}^{x} s(t,\tilde{x})\,\mathrm{d}\tilde{x} \tag{2.5}$$

and

$$\varrho(\xi,t) := s(x,t). \tag{2.6}$$

Theorem 2.1. $\varrho(\xi,t)$ defines a solution of the HD equation on the interval $[0, 2\sqrt{c}]$.

A direct proof of this statement is possible; however it involves tedious computations. The result is, up to the influence of the spectral parameter c, the one obtained in [11], so a direct proof will, more or less, follow the lines exhibited in that paper. It was found that the spectral parameter has a definite group theoretical meaning. So, instead of giving a direct proof, we will show how this result follows by taking into account the known facts from the group analysis of the equations under consideration. When u is a multi-soliton solution of the KdV, we exhibit, later, a more simplified method for performing the necessary computations in order to find explicit solutions.

3. Justification of the method

It is well known [3, 1] (see also [6]) that

$$\tilde{u} + u + c\mathrm{D}^{-1}(\tilde{u} - u) + \tfrac{1}{2}(\mathrm{D}^{-1}(\tilde{u} - u))^2 = 0 \tag{3.1}$$

constitutes an auto-Bäcklund transformation of the KdV. Here, as usual, D^{-1} denotes the inverse of differentiation with respect to x. This means that, if u is a solution of the KdV, then $\tilde{u}$ is another solution (one where a soliton with speed c has been added). Furthermore, we know that when s_x is an eigenvector, with eigenvalue c, of the recursion operator of the KdV

$$\phi(\tilde{u}) = \mathrm{D}^2 + 2\tilde{u} + 2\mathrm{D}\tilde{u}\mathrm{D}^{-1}$$

then s_x must be in the kernel of the variational derivative of (3.1) with respect to $\tilde{u}$ (see [6], [7] and [2]). To be precise, we then have

$$s_x + cs + s\mathrm{D}^{-1}(\tilde{u} - u) = 0. \tag{3.2}$$

From this equation we can express $\mathrm{D}^{-1}(\tilde{u} - u)$ in terms of s and insert that result in (3.1). This leads to

$$\mathrm{D}^{-1}(\tilde{u} - u) = -\left(\frac{s_x}{s} + c\right)$$

and

$$2u - \left(\frac{s_x}{s}\right)_x + \frac{1}{2}\left(\frac{s_x}{s} + c\right)^2 - c\left(\frac{s_x}{s} + c\right) = 0. \tag{3.3}$$

If we define

$$\varphi = s_x/s \tag{3.4}$$

then we obtain (2.2). This consideration shows that the function s, which we obtain from equation (2.3), must be the integral of the eigenvector of $\Phi(\tilde{u})$, with eigenvalue c i.e. an eigenvector of the recursion operator of the field $\tilde{u}$, which arises when one soliton with speed c is added to the field u. It is well known that the dynamics of an eigenvector of the recursion operator is the same as the dynamics of an infinitesimal generator of a one-parameter symmetry group [5]. Hence we can rewrite the u-dynamics from (2.1) into a dynamical law for the variable s. This (see [7]) leads to the following PDE

$$s_t = s_{xxx} - \frac{3s_x s_{xx}}{s} + \frac{3s_x^2}{2s^2} + \frac{3}{2}cs_x. \tag{3.5}$$

If $s = \phi_x$ is inserted, then this gives, up to the additional term $(\frac{3}{2})cs_x$, the well known dynamics of the singularity manifold equation of the KdV [30] (see also [8]). Another well known fact of Painlevé analysis [23] is that a suitable reciprocal transformation leads from the singularity manifold equation to the HD equation. In addition, we know ([8] or [4]) that all generators of the translation group are annihilated by the reciprocal transformation given by (2.6). Hence, putting all these transformations together, we indeed arrive, with the reciprocal transformation of s, at a solution of the HD equation.

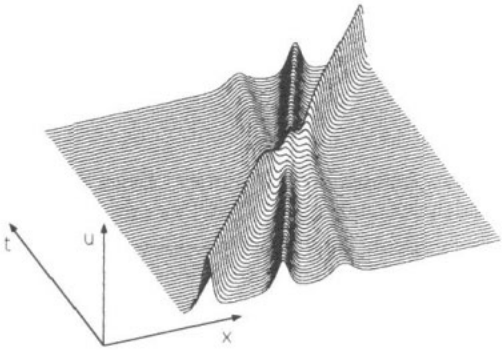

Figure 1. Three-soliton of the KdV: $c_1 = 0.4$; $c_2 = 0.6$; $c3 = 0.8$; and $-10 < x < 10$, $-10 < t < 10$, $0 < u < 6$.

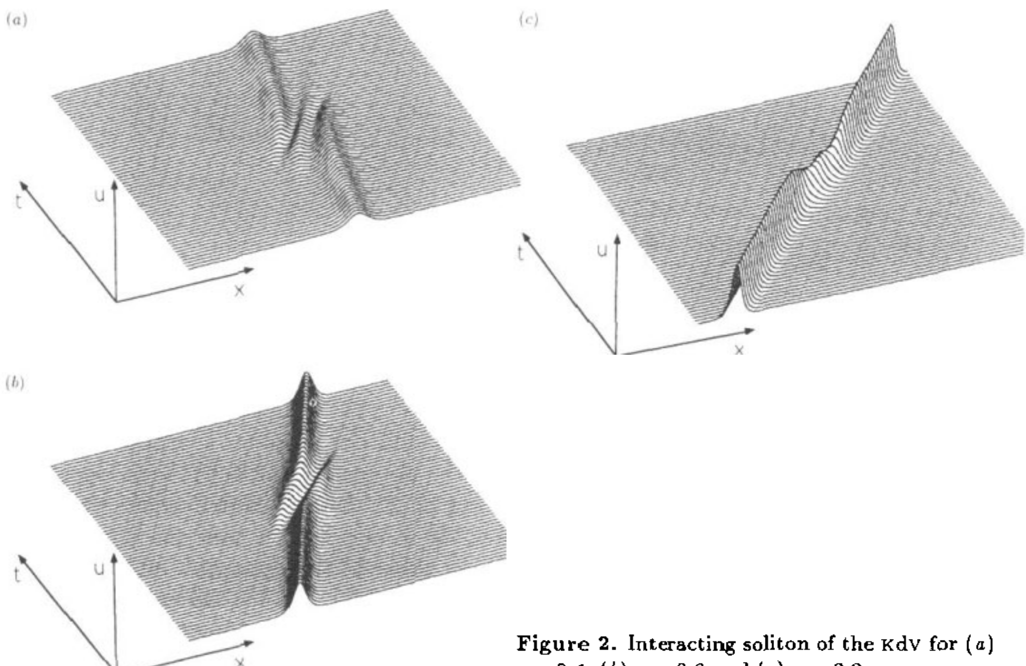

Figure 2. Interacting soliton of the KdV for (*a*) $c = 0.4$, (*b*) $c = 0.6$ and (*c*) $c = 0.8$.

4. The multi-soliton case

However, the method presented in the last section still does not give us explicit solutions directly since a Riccati equation (2.2) has to be solved. However, this Riccati equation can be solved explicitly in special cases, among others, when u is a multi-soliton solution of the KdV.

Instead of going through the necessary computations, we will present here an abbreviated method for that case. We start with an arbitrary N-soliton solution

$\tilde{u}_N(x,t)$ of the KdV. Such a solution is explicitly obtained, for example, by the Hirota bilinear formula [1]. We parametrize this solution by its asymptotic speeds $(c_1, \ldots, c_N)$ and its phases $(q_1, \ldots, q_N)$. Then we know ([9] and [20]) that the partial derivative of $(\tilde{u}_N)_x$, with respect to any of the q_i, gives eigenvectors of the recursion operator with corresponding eigenvalues c_i. These eigenvectors coincide with the result of (2.3), where the normalizion had been fixed by a suitable choice of the constant of integration. So we have

$$s_{i_x} = \partial(\tilde{u}_N)_x/\partial q_i. \tag{4.1}$$

Now, performing the reciprocal transformation (2.6), we directly obtain the desired solution of the HD equation. Even if the reciprocal transformation may be difficult to handle from the analytic viewpoint, it is a very simple operation from the point of view of plotting these solutions, since all one has to do is to use parametric plots. If one does that in case of a three-soliton solution of the KdV (figure 1) with asymptotic speeds $c_1 = 0.4$, $c_2 = 0.6$, $c3 = 0.8$ one obtains, by taking partial derivatives with respect to the three phases q_1, q_2, q_3, the plots (figure 2) for the interacting solitons s_i, $i = 1, 2, 3$.

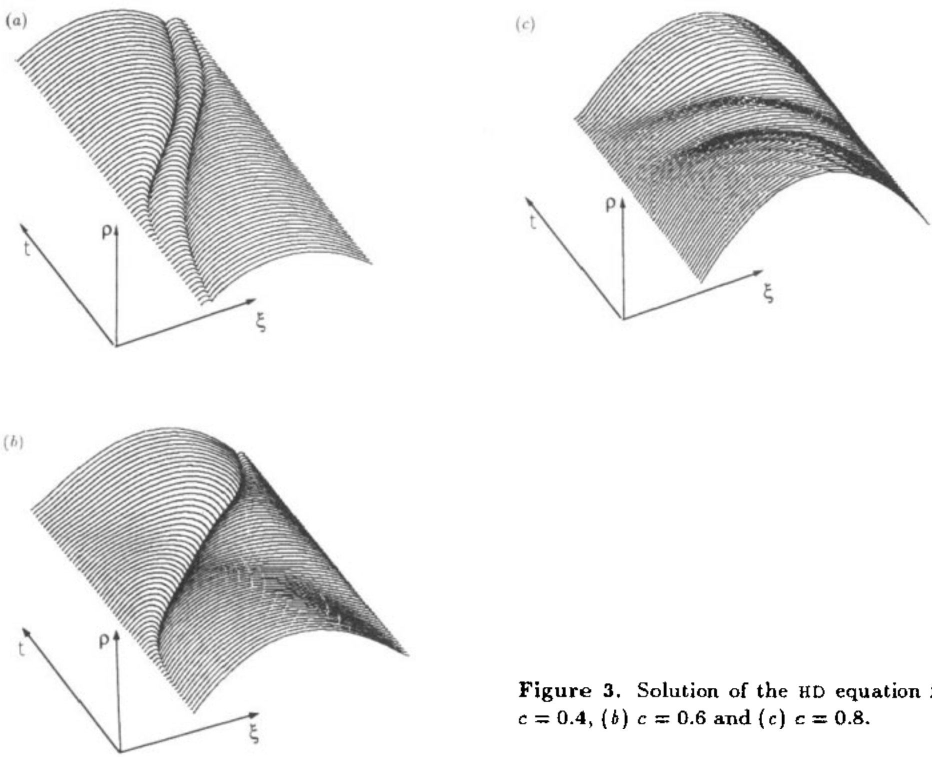

Figure 3. Solution of the HD equation for (a) $c = 0.4$, (b) $c = 0.6$ and (c) $c = 0.8$.

By application of the reciprocal transformation we then find the corresponding solutions of the HD equation (figure 3).

A look at these plots shows where some of the computational difficulties with the HD equation arise. Since the dents, which are the results of additional solitons added to a one-soliton of the KdV, disappear exponentially for $t \to \pm\infty$ it becomes quite clear that, in general, the solution of the HD equation is not stable with respect to initial data because small changes can grow exponentially.

5. Concluding remarks

We have already given one solution of this kind for the HD equation in [8], therefore we conclude this paper by pointing out its novel features. First of all, we have shown how, in general, by use of Riccati transcendants, KdV solutions can be transformed into HD solutions by simple methods. In [8] this was only possible for multi-solitons. Here, the method is shown to be of general validity. We needed multi-soliton solutions for the abbreviated computation. In [8] we still had to solve the eigenvector problem for the recursion operator in order to arrive at the interacting soliton s. Here, we only have to compute a simple derivative for functions which are well known from the literature. Thus the computation of those explicit solutions for HD, which are related to KdV multi-solitons, has been trivialized.

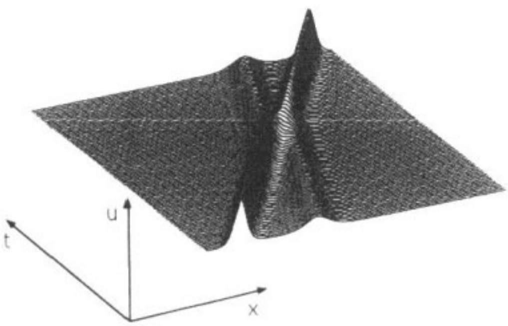

Figure 4. Two-soliton of the CDGSK: $c_1 = 0.5$; $c_1 = 1.0$; $c3 = 0.8$ with $-13 < x < 13$, $-15 < t < 15$, $0 < u < 6$.

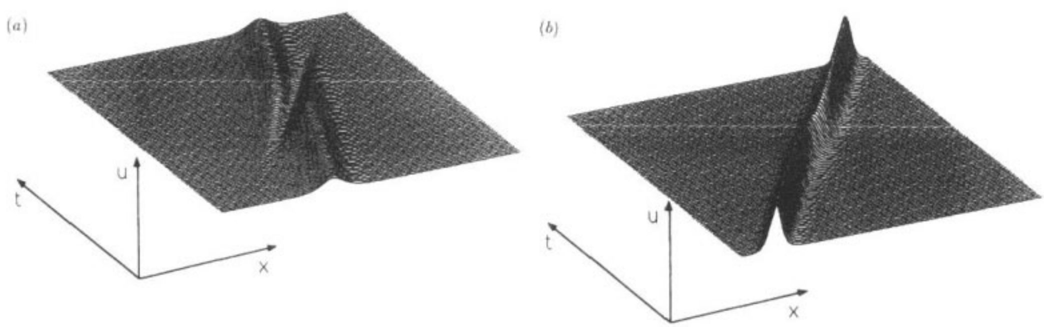

Figure 5. Interacting soliton of the CDGSK: (a) $c = 0.5$ and (b) $c = 1.0$.

It is interesting whether or not by using the same method multi-solitons of, say, the Caudrey–Dodd–Gibbon–Sanada–Kotera (CDGSK) equation can be transformed into solutions of its reciprocal counterpart [14], i.e. the Kamamoto equation [14, 4]

$$\varrho_t = 10\varrho^4\varrho_{xx}\varrho_{xxx} + 5\varrho^4\varrho_x\varrho_{xxxx} + \varrho^5\varrho_{xxxxx}. \tag{5.1}$$

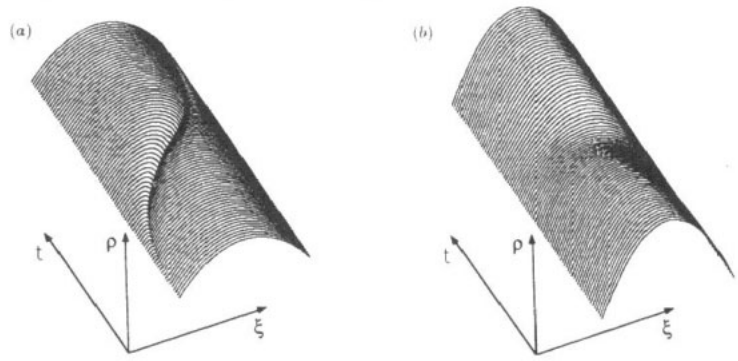

Figure 6. Solution of the Kawamoto equation (a) $c = 0.5$ and (b) $c = 1.0$.

This question is not completely trivial. Although we know that in this case a reciprocal transformaton of the singularity manifold equation leads to the Kawamoto equation, it is not that easy to see whether the additional terms by which the singularity equation differs from the interacting soliton equation are cancelled by the reciprocal transformation. However, a detailed analysis shows that this is indeed the case, and that the method goes through without any change. Explicit solutions from this procedure look very much like those for the HD case. For example (figure 4), when a two-soliton for the CDGSK is undergoing a similar sequence of transformations as those described in this paper, we first obtain interacting solitons of the CDGSK (figure 5). And then, by reciprocal transformation, solutions of the Kawamoto equation as given in figure 6.

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